

Semidiscrete evolution of elastic waves in a piezoelectric solid

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Abstract

We consider a model problem of the propagation of elastic waves which are coupled with an electric field inside a piezoelectric solid as well as the discretization of this problem in space. The stress tensor in the solid combines the effect of a linear dependence of strains with the influence of an existing electric field. The system is closed using Gauss's law for the associated electric displacement. We use a first order in time and space differential system to study the well-posedness of both problems. This requires the use of an elliptic lifting operator. In the semidiscrete case we formulate the problem corresponding to an abstract Finite Element discretization in the electric and elastic fields.

Keywords: Piezoelectricity, finite elements, elastic wave propagation

1 Introduction

We consider a solid occupying a bounded region of space $\Omega \subset \mathbb{R}^3$. At positive time an elastic wave is triggered in the solid. Formally the equations we consider are the following

$$\begin{aligned} \rho \mathbf{u}_{tt} &= \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} & \Omega \times [0, \infty), \\ \operatorname{div} \mathbf{D} &= 0 & \Omega \times [0, \infty), \end{aligned}$$

where $\mathbf{u}$ represents the elastic displacement, $\boldsymbol{\sigma}$ the stress, $\mathbf{D}$ the electric displacement, ρ the mass density, and $\mathbf{f}$ any source terms. Expressing the electric field through an electric potential, ψ , the piezoelectric behavior of the solid is expressed in two ways. The first is through the definition of stress as the combination of an instantaneous linear operator, $\mathcal{C}$, acting on strain (Hooke's Law) and the effect of electric fields in the solid:

$$\boldsymbol{\sigma} := \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{e}\nabla\psi, \quad (1)$$

where $\boldsymbol{\varepsilon}(\mathbf{u})$ is the symmetric strain tensor and $\mathbf{e}$ is the third-order piezoelectric tensor. We can also see the piezoelectric effect through the definition of the electric displacement for which we

imposed Gauss' Law above,

$$\mathbf{D} := \mathbf{e}^\top \boldsymbol{\varepsilon}(\mathbf{u}) - \kappa \nabla \psi, \quad (2)$$

where κ is the dielectric tensor. We consider two partitions of the boundary $\Gamma = \partial\Omega$ into relatively open sets such that

$$\Gamma_{\text{ds}} \cap \Gamma_{\text{tr}} = \Gamma_{\text{pt}} \cap \Gamma_{\text{fl}} = \emptyset,$$

and

$$\Gamma = \overline{\Gamma_{\text{ds}}} \cup \overline{\Gamma_{\text{tr}}} = \overline{\Gamma_{\text{pt}}} \cup \overline{\Gamma_{\text{fl}}}.$$

The partitions are independent of one another. We consider the following boundary conditions as part of our model problem:

$$\begin{aligned} \gamma \mathbf{u} &= \mathbf{g} & \Gamma_{\text{ds}} \times [0, \infty), \\ \boldsymbol{\sigma} \boldsymbol{\nu} &= \mathbf{h} & \Gamma_{\text{tr}} \times [0, \infty), \\ \gamma \psi &= \mu & \Gamma_{\text{pt}} \times [0, \infty), \\ \mathbf{D} \cdot \boldsymbol{\nu} &= \eta & \Gamma_{\text{fl}} \times [0, \infty). \end{aligned}$$

Here γ represents the trace operator and $\boldsymbol{\nu}$ the outward pointing unit normal vector on Γ . Finally we require the initial conditions

$$\mathbf{u} = \mathbf{u}_t = \mathbf{0} \quad t = 0,$$

which signifies that at the initial time the solid is at rest. We will treat this system in the way of abstract evolution equations where our unknowns are mappings in the time variable to the appropriate Hilbert space, which is where we deal with the space differentiation. In this way we can consider our unknowns $\mathbf{u}(t)$ and $\psi(t)$ to be elements of subsets of $H^1(\Omega)^3$ and $H^1(\Omega)$ respectively, for each $t \geq 0$.

2 Semidiscrete formulation

The semidiscrete version of the problem will be formulated with the intention of using finite elements to solve for the elastic displacement and electric potential. To this end we set finite dimensional subspaces $\mathbf{V}_h \subset H^1(\Omega)^3$ and $W_h \subset H^1(\Omega)$ to be our generic finite element spaces. We also consider the spaces

$$\begin{aligned} \mathbf{V}_h^0 &:= \{\mathbf{u}^h \in \mathbf{V}_h : \gamma \mathbf{u}^h = 0 \text{ on } \Gamma_{\text{ds}}\}, \\ W_h^0 &:= \{\psi^h \in W_h : \gamma \psi^h = 0 \text{ on } \Gamma_{\text{pt}}\}, \end{aligned}$$

and interpolation operators

$$\begin{aligned}\mathbf{I}_h : H^{1/2}(\Gamma_{\text{ds}})^d &\longrightarrow \gamma \mathbf{V}_h|_{\Gamma_{\text{ds}}}, \\ I_h : H^{1/2}(\Gamma_{\text{pt}}) &\longrightarrow \gamma W_h|_{\Gamma_{\text{pt}}}.\end{aligned}$$

Using parentheses to represent the $L^2(\Omega)^3$ and $L^2(\Omega)^{3 \times 3}$ inner products and angle brackets to represent the duality pairings of $H^{-1/2}$ and $H^{1/2}$ on appropriate parts of the boundary Γ we state the semidiscrete problem in variational form as looking for $(\mathbf{V}_h \times W_h)$ -valued $(\mathbf{u}^h, \psi^h)$ which for all $t \geq 0$ satisfy

$$\begin{aligned}(\rho \mathbf{u}_{tt}^h, \mathbf{w}) + (\sigma^h, \varepsilon(\mathbf{w})) &= (\mathbf{f}, \mathbf{w}) + \langle \mathbf{h}, \gamma \mathbf{w} \rangle_{\Gamma_{\text{tr}}} \\ &\quad \forall \mathbf{w} \in \mathbf{V}_h^0, \\ (\mathbf{D}^h, \nabla \phi) &= \langle \eta, \gamma \phi \rangle_{\Gamma_{\text{R}}} \quad \forall \phi \in W_h^0, \\ \gamma \mathbf{u}^h &= \mathbf{I}_h \mathbf{g} \quad \text{on } \Gamma_{\text{ds}}, \\ \gamma \psi^h &= I_h \mu \quad \text{on } \Gamma_{\text{pt}},\end{aligned}$$

where for brevity we have used the symbols σ^h and $\mathbf{D}^h$ which are defined in terms of the semidiscrete unknowns $\mathbf{u}^h$ and ψ^h exactly as in (1) and (2).

3 First Order Form

The mathematical analysis of both the continuous and semidiscrete problems is undertaken by rewriting them into a first order system of the form

$$\begin{aligned}\dot{U}(t) &= A_* U(t) + F(t), \\ BU(t) &= \xi(t), \\ U(0) &= 0,\end{aligned}$$

for a certain operator A_* that involves a first order in space differential operator and the inverse of an elliptic operator and boundary operator B . The unknowns collected in U are related to displacement, purely elastic stress and electric field, while the terms collected in F correspond to source terms and Neumann boundary conditions and ξ to Dirichlet boundary conditions. In this form, we follow the template of [2] to arrive at stability bounds and error estimates in the time domain. This technique yields sharper estimates than the Laplace domain technique found in [4] for a similar problem, which also includes acoustic coupling across Γ .

4 Extensions

With the introduction of an incident acoustic wave in Ω^c , we can extend the problem outlined above into the a wave-structure interaction problem, where in addition to solving for

the elastic displacement and electric potential, we look for a scattered acoustic field outside the solid. In this case, the problem is formulated as in [1]. When solving for the acoustic unknowns, we use a retarded potential representation and discretize using boundary elements. Another easy extension of this problem arises when we note that in Ω , the tensor $\mathbf{e}$ serves to couple the quantities $\mathbf{u}$ and ψ . If we take $\mathbf{e} \equiv 0$, the system is reduced to a wave-structure interaction problem (if we include acoustics) with a purely elastic solid. The stability and error analysis we obtain for the piezoelectric problem also apply to this problem and can be compared to their Laplace domain counterparts in [3].

References

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