FEM-BEM coupling for transient acoustic scattering by thermoelastic obstacles

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Abstract

We present a combined field and boundary integral equation method for solving time-dependent scattering of acoustic waves by thermoelastic obstacles. The approach is geared towards a finite element discretization in the interior of the scatterer and a boundary element approximation of the acoust field. Using an integral representation of the solution in the infinite exterior domain occupied by the fluid, the problem is reduced to one defined only over the finite region occupied by the solid, with nonlocal boundary conditions. The resulting non local boundary problem is analized using the Laplace transform in terms of time-domain data. Existence and uniqueness results are established in the Laplace domain where Galerkin semi-discretization approximations are derived and error estimates are obtained. Full space-time discretization and time-domain error estimates based on the Convolution Quadrature method are also presented.

Keywords: Fluid-structure interaction, Coupling BEM-FEM, Time-domain boundary integral equation, Wave scattering, Convolution quadrature.

1 Introduction

We are concerned with a time-dependent direct scattering problem in fluid-thermoelastic solid interaction, which can be simply described as follows: an acoustic wave propagates in a fluid domain of infinite extent in which a bounded thermoelastic body is immersed. We denote Ω_- the bounded domain occupied by the thermoelastic body with a Lipschitz boundary Γ and we let $\Omega_+ := \mathbb{R}^3 \setminus \overline{\Omega}_-$ be its exterior. The problem is then to determine the scattered velocity potential V in the fluid domain, the deformation of the solid $\mathbf{U}$ and the variation of the temperature Θ in the obstacle with respect to the equilibrium temperature Θ_0.

In the linear small displacement regime, the governing equations are [3] :

$$\begin{aligned} c^{-2}\partial_{tt}V - \Delta V &= 0 \quad \text{in } \Omega_+ \times (0,T), \\ \rho_\Sigma \partial_{tt}\mathbf{U} - \Delta^*\mathbf{U} + \zeta\,\nabla\Theta &= \mathbf{0} \quad \text{in } \Omega_- \times (0,T), \\ \kappa^{-1}\partial_t\Theta - \Delta\Theta + \eta\partial_t(\nabla\cdot\mathbf{U}) &= 0 \quad \text{in } \Omega_- \times (0,T), \end{aligned}$$

where ρ_Σ is the density of the solid, κ is the thermic difussivity, η is a constant proportional to the thermal expansion coefficient ζ. As usual the symbol Δ^* is the Lamé operator defined by

$$\Delta^*\mathbf{U} := \mu\Delta\mathbf{U} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{U}).$$

On the interface Γ between the solid and the fluid we have the transmission conditions

$$\begin{aligned} (\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{U}) - \zeta\Theta\mathbf{I})\mathbf{n} &= -\,\rho_f\partial_n(V + V^{inc})\mathbf{n}, \\ \partial_t\mathbf{U}\cdot\mathbf{n} &= -\,\partial_n(V + V^{inc}) \\ \partial_n\Theta &= 0 \end{aligned}$$

where $\mathbf{n}$ is the exterior unit normal to Ω_-, and V^{inc} denotes the given incident field. We assume the causal initial conditions

$$\begin{aligned} \mathbf{U}(x,t) &= \frac{\partial\mathbf{U}(x,t)}{\partial t} = \mathbf{0}, \quad \Theta(x,t) = 0, \\ V(x,t) &= \frac{\partial V}{\partial t}(x,t) = 0. \end{aligned}$$

2 An integro-differential system

Following the ideas introduced in the seminal work of Bamberger and Ha-Duong [1] the system is transformed into the Laplace domain and recast as an integro-differential problem making use of a layer potential representation of the acoustic field.

The resulting non-local system below involves the operators of the Calderón projector for the acoustic field but retains the weak PDE formulation for the elastic and thermal unknowns. It can be represented in matrix operator form as

$$\mathbf{L}(\mathbf{u},\theta,\phi,\lambda) = (d_1,d_2,d_3,d_4),$$

where the data (d_1,d_2,d_3,d_4) is given by

$$\begin{aligned} d_1 &= -\,s\;\rho_f\;\gamma^{-\prime}(\gamma^+ v^{inc}\mathbf{n}), \quad & d_2 &= 0, \\ d_3 &= \partial_n^+ v^{inc}, & d_4 &= 0, \end{aligned}$$

and the matrix of integral and differential operators is given by

$$\mathbf{L} := \begin{pmatrix} \mathbf{A}_s & -\zeta(div)' & s\rho_f\, \gamma^{-\prime}\mathbf{n} & 0 \\ s\eta\; div & \mathrm{B}_s & 0 & 0 \\ -s\mathbf{n}^\top\gamma^- & 0 & \mathrm{W}(s) & -(\frac{1}{2}\mathrm{I}-\mathrm{K}(s))' \\ 0 & 0 & \frac{1}{2}\mathrm{I}-\mathrm{K}(s) & \mathrm{V}(s) \end{pmatrix}.$$

Above $s \in \mathbb{C}_+$ is the Laplace parameter and

$$\mathbf{A}_s : \mathbf{H}^1(\Omega_-) \to (\mathbf{H}^1(\Omega_-))',$$
$$\mathrm{B}_s : H^1(\Omega_-) \to (H^1(\Omega_-))'$$

are the operators associated with the elastic and thermal bilinear forms

$$a(\mathbf{u},\cdot;s) := (\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\cdot))_{\Omega_-} + s^2\rho_\Sigma(\mathbf{u},\cdot)_{\Omega_-},$$
$$b(\theta,\cdot;s) := (\nabla\theta,\nabla(\cdot))_{\Omega_-} + (s/\kappa)(\theta,\cdot)_{\Omega_-}.$$

3 Continuous and discrete well posedness

We study the conforming Galerkin discretization of the system on arbitrary closed subspace of the solution space endowed with the appropriate energy norm. The well posedness of the discrete problem is done by reformulating this equations in terms of an exotic transmission problem and showing the coercivity of the resulting operator. This techinique is similar to the one discussed in [4] for purely acoustic/elastic interaction.

Error estimates for the semidiscretization are obtained by observing that an appropriate elliptic projection of the error term satisfies equations with the same structure as the ones satisfied by the unknowns, to which all of the above machinery can be applied.

Bounds are established explicitly in terms of the laplace parameter s and its real part σ. This enables to translate the estability estimates into the time-domain while mantaiing explicit knowledge of the behaviour of the bounding constants with respect to time. The fact that the analysis is carried out in arbitrary closed subspaces implies that, by taking the discrete space to be the entire solution space, the well posedness of the continuous problem is obtained as a by-product.

4 A fully discrete system

The linear system arising from the discretization has a structure that can be depicted by the block matrix

$$\begin{bmatrix} \mathbf{FEM}(s) & s\rho_f(\mathrm{N}\Gamma)_h^t \\ -s\rho_f(\mathrm{N}\Gamma)_h & \mathbf{BEM}(s) \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathbf{u}^h \\ \theta^h \end{bmatrix} \\ \begin{bmatrix} \lambda^h \\ \phi^h \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -s\rho_f\Gamma_h^t\beta^h \\ \eta^h \end{bmatrix} \\ \begin{bmatrix} 0 \\ \rho_f\alpha^h \end{bmatrix} \end{bmatrix},$$

where the sparse Finite Element block contains mass and stiffness matrices as well as first order terms related to the elastic and thermal unknowns. The boundary element block $\mathbf{BEM}(s)$ contains the Galerkin discretization of the operators of the acoustic Calderón calculus and the coupling trace matrix $(\mathrm{N}\Gamma)_h$ is the discretization of the bilinear form arising from the duality pairing $\langle \mathbf{u}^h\cdot\boldsymbol{\nu},\chi^h\rangle_\Gamma$.

The system is discretized in time using BDF2 Convolution Quadrature [5] for the integral equation block and BDF2 time-stepping for the finite element block. As pointed out in [2], this approach is equivalent to an application of BDF2-CQ to the entire discrete matrix. This fact is used to derive error estimates for the discretization by considering a global CQ approximation-resulting in a second order accurate scheme. Numerical convergence studies and simulations for the 2D case are carried out.

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