

A system of boundary integral equations for transient wave-structure interaction

Tonatiuh Sánchez-Vizuet^{1,*}, Francisco-Javier Sayas²

¹Department of Mathematical Sciences, University of Delaware, Newark, DE , USA

²Department of Mathematical Sciences, University of Delaware, Newark, DE , USA

*Email: tonatiuh@udel.edu

Abstract

In this paper we present a system of time-domain boundary integral equations describing the scattering of acoustic waves by an elastic obstacle. The analysis of well-posedness is done simultaneously at the continuous and semidiscrete level. We use Laplace domain techniques for the analysis of well-posedness and for the study of the error due to Galerkin semidiscretization of the integral system.

Keywords: boundary integral equations, wave-structure interaction, vector-valued distributions

1 The differential system

We will consistently use evolution equation notation for our partial differential equations. Like this, a function of space-and-time variables will be thought as a function of time with values in a vector space of functions of the space variables. Differentiation with respect to the space variables will be written with the usual symbols of vector calculus (gradient, divergence, laplacian) without making any reference to the space variables. Differentiation with respect to time will be written with an upper dot. Time-domain integral operators and potentials will be written in convolutional notation, emphasizing this same effect: considered as vector- or operator-valued distributions of the time variable, they correspond to causal convolution of distributions of a single variable.

Consider a bounded Lipschitz domain (possibly non-connected) $\Omega_- \subset \mathbb{R}^d$ with boundary Γ and exterior Ω_+ . Let us assume that a linearly elastic material, subject to small deformations, occupies Ω_- . We characterize the material properties of this object by its two Lamé parameters and its mass density. This means that the displacement field satisfies the elastic wave equation

$$\rho_s \ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}), \quad (1)$$

where

$$\boldsymbol{\sigma}(\mathbf{u}) = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) + \lambda(\nabla \cdot \mathbf{u}) \mathbf{I}$$

is the stress tensor for a Hookean material. The normal traction on Γ is denoted $\mathbf{t}(\mathbf{u}) = \boldsymbol{\sigma}(\mathbf{u})\boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is the unit normal vector field on Γ pointing from Ω_- to Ω_+ .

An acoustic field is measured on a fluid surrounding Ω_- . The interaction [3] will be triggered by a given incident wave v^{inc} . The total wave $v^{\text{tot}} = v + v^{\text{inc}}$ is decomposed as the sum of the incident wave and a scattered wave, which satisfies the acoustic wave equation

$$c^{-2} \ddot{v} = \Delta v \quad (2)$$

and a radiation-causality condition that can be expressed in simple terms: at every time $t > 0$ the support of v is contained in a ball of radius that grows proportionally to ct . The interaction is mathematically represented by two transmission conditions:

$$\dot{\mathbf{u}} + \nabla v^{\text{tot}} \cdot \boldsymbol{\nu} = 0 \quad (3)$$

and

$$\mathbf{t}(\mathbf{u}) + \rho_f \dot{v}^{\text{tot}} \boldsymbol{\nu} = \mathbf{0}. \quad (4)$$

The parameters c and ρ_f are the speed of waves and density of the acoustic fluid. At time $t = 0$ we assume that the incident wave has not reached the elastic obstacle and therefore the solid is at rest and the only acoustic field is the incident one.

The simplest approach to prove that (1)-(4) is well posed consists of taking Laplace transforms of the equations and write a variational formulation [1]. Assuming that v^{inc} is a tempered causal distribution with values in

$$H_\Delta^1(\mathcal{O}) := \{v \in H^1(\mathcal{O}) : \Delta v \in L^2(\mathcal{O})\},$$

where $\mathcal{O}$ is an open neighborhood of Γ , we can prove that (1)-(4) has a unique solution $(\mathbf{u}, v)$ which is a causal Laplace transformable distribution with values in $\mathbf{H}_\sigma^1(\Omega_-)^3 \times H_\Delta^1(\Omega_+)$, where

$$\mathbf{H}_\sigma^1(\Omega_-) := \{\mathbf{u} \in H^1(\Omega_-)^3 : \boldsymbol{\sigma}(\mathbf{u}) \in L^2(\Omega_-)^{3 \times 3}\}.$$

2 An equivalent integral system

The layer potentials associated to the acoustic wave equation around Γ can be defined as determining the unique causal $H_{\Delta}^1(\mathbb{R}^d \setminus \Gamma)$ -valued solution $v = \mathcal{S} * \eta - \mathcal{D} * \phi$ of the wave equation

$$c^{-2}\ddot{v} = \Delta v$$

satisfying the transmission conditions

$$\begin{aligned}\gamma^- v - \gamma^+ v &= \phi, \\ \partial_\nu^- v - \partial_\nu^+ v &= \eta,\end{aligned}$$

where the superscripts $\pm$ are used to tag traces and normal derivatives from $\Omega_{\pm}$. Four retarded integral operators can be defined as follows:

$$\begin{aligned}\mathcal{V} * \eta &:= \gamma^-(\mathcal{S} * \eta) = \gamma^+(\mathcal{S} * \eta), \\ \mathcal{K} * \phi &:= \frac{1}{2}\gamma^-(\mathcal{D} * \phi) + \frac{1}{2}\gamma^+(\mathcal{D} * \phi), \\ \mathcal{J} * \eta &:= \frac{1}{2}\partial_\nu^-(\mathcal{S} * \eta) + \frac{1}{2}\partial_\nu^+(\mathcal{S} * \eta), \\ \mathcal{W} * \phi &:= -\partial_\nu^-(\mathcal{D} * \phi) = -\partial_\nu^+(\mathcal{D} * \phi).\end{aligned}$$

Definitions in very much the same spirit can be given for the elastic retarded layer potentials and operators, which we will denote with the same letters in boldface. We finally need two operators related to the normal vector field:

$$\mathbf{N}\phi := \phi \cdot \boldsymbol{\nu}, \quad \mathbf{N}^t \eta := \eta \boldsymbol{\nu}.$$

This work relates the system (1)-(4) with the following system of integral equations:

$$\begin{aligned}\mathbf{W} * \phi + \rho_f \mathbf{N}^t \mathcal{V} * (\mathbf{N}\ddot{\phi}) \\ + \rho_f (\mathbf{N}^t (\mathcal{K} * \dot{\phi}) - \mathcal{J} * (\mathbf{N}\dot{\phi}))\end{aligned}\tag{5}$$

$$\begin{aligned}&= -\rho_f \mathbf{N}^t \partial_\nu \dot{v}^{\text{inc}} \\ &\quad + \rho_f (-\frac{1}{2} \mathbf{N}^t \gamma \dot{v}^{\text{inc}} + \mathcal{J} * (\mathbf{N}^t \gamma \dot{v}^{\text{inc}})) \\ \rho_f (\mathbf{N}(\mathcal{K} * \dot{\phi}) - \mathcal{J} * (\mathbf{N}\dot{\phi})) \\ &+ \rho_f^2 \mathbf{N} \mathcal{V} * (\mathbf{N}^t \ddot{\phi}) + \rho_f \mathcal{W} * \phi \\ &= \rho_f (\frac{1}{2} \partial_\nu v^{\text{inc}} + \mathcal{J} * \partial_\nu v^{\text{inc}}) \\ &\quad - \rho_f^2 \mathbf{N} \mathcal{V} * (\mathbf{N}^t \gamma \ddot{v}^{\text{inc}})\end{aligned}\tag{6}$$

In this work we prove that the wave-structure interaction problem (1)-(4) is equivalent to the system (5)-(6) in the sense that the unknowns of the latter are related to the values of the former by

$$\phi = \gamma^- \mathbf{u}, \quad \phi = \gamma^+ v.$$

We also show that the system (5)-(6) admits a unique solution, even after Galerkin semidiscretization in space with any discrete pair $\mathbf{Y}_h \times$

$\mathbf{Y}_h \subset H^{1/2}(\Gamma)^d \times H^{1/2}(\Gamma)$. The study also includes mapping properties for the solution of (5)-(6) and the reconstruction of the solution of (1)-(4) using integral representation formulas. It also incorporates the study of the effect of semidiscretization in space. This is done with techniques developed in a systematic way in [4], but going back to the seminal work [1]. Following [2], the analysis leaves the full discretization with Lubich's Convolution Quadrature ready to be applied and analyzed.

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